# Analytic framework

The analytic framework includes equations from the atmospheric fluid dynamics and wind farm fluid dynamics. The equations from atmospheric fluid dynamics concern steady-state, horizontal, large-scale flows under neutral conditions above an infinite homogeneous surface with roughness [1][2][3]. First, we recall the governing equations of the geostrophic flow, i.e., the momentum equations [4]:

where and are the geostrophic wind components ( is the modulus), the pressure, the density, and the Coriolis parameter. The Coriolis parameter is given by , where is the rotation rate of the Earth (7.2921∙10−5 rad/s), and the latitude. In the Ekman layer, mechanical turbulence becomes important, and the resulting momentum equations then become [5]:

where and are the mean wind components, and and are the turbulent fluxes (also called Reynolds stresses). Substituting the pressure-gradient terms of Eq. 2 with the Coriolis force terms from Eq. 1, the equations of motion can be written as [5]:

From these governing equations, Blackadar and Tennekes [1] derived, for the case of large Rossby numbers, , two kinds of self-similar solutions, one valid only in the Ekman layer well outside the surface layer and another valid inside the surface layer [3]. By matching those solutions in a region of overlap, they derived the following expressions:

where is the surface friction velocity and the Von Kármán constant (≈ 0.4). The resistant constants (4 and 12) on the right-hand side of Eq. 4 are derived in Refs. [1][3]. These equations relate the large-scale flow (geostrophic flow and Ekman layer) with the small-scale flow (surface layer, where turbines are located).

The equations from wind farm fluid dynamics concern instead steady-state, horizontal flows under neutral conditions within and over an infinite wind farm [6][7]. In this case, the derivation neglects any effect of the Coriolis force and focuses therefore on smaller spatial scales. This analysis provides an equivalent surface roughness, , for the effect of a large wind farm on the overlying atmospheric boundary layer [6]:

where and are the turbine diameter and huh height, respectively, the actual surface roughness of the underlying terrain, and

where is the hub-height wind speed, the equivalent surface friction velocity generated by the wind farm, is the thrust coefficient provided by the manufacturer (see Supplementary Fig. 1), and and are the horizontal nondimensional spacings. For the case of aligned layout and installed capacity density of 9 W/m2, , whereas for the case of staggered layout and installed capacity density of 4.5 W/m2, .

The hub-height wind speed is calculated according to [6]:

Here, we combine Eq. 5 with Eqs. 6-9 to provide a closed-form system that relates the influence of geostrophic wind and Coriolis parameter (latitude-dependent) with the wind farm power density. and in Eq. 5 are replaced with and to account for the effect of the wind farm on the Ekman layer. The system represented by Eqs. 5-9, with unknowns , , , and , can be solved iteratively and provides a value for , with which we calculate the power produced according to the power curve given by the manufacturer.

# References

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